Reflectionless Potentials and $\mathcal{PT}$ Symmetry

Zafar Ahmed[1], Carl M Bender[2], and M V Berry[3]

[1]Nuclear Physics Division, Bhabha Atomic Research Centre, Trombay,
    Mumbai 400 085, India
[2]Department of Physics, Washington University, St. Louis MO 63130, USA
[3]H. H. Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, UK

Abstract.   Large families of Hamiltonians that are non-Hermitian in the conventional sense have been found to have all eigenvalues real, a fact attributed to an unbroken $\mathcal{PT}$ symmetry. The corresponding quantum theories possess an unconventional scalar product. The eigenvalues are determined by differential equations with boundary conditions imposed in wedges in the complex plane. For a special class of such systems, it is possible to impose the $\mathcal{PT}$-symmetric boundary conditions on the real axis, which lies on the edges of the wedges. The $\mathcal{PT}$-symmetric spectrum can then be obtained by imposing the more transparent requirement that the potential be reflectionless.

The purpose of this letter is to point out a connection between real reflectionless potentials and quantum mechanics defined by complex $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian operators.

Research begun in 1998 [1, 2, 3, 4] has established that the eigenvalues of the complex ϵ-deformed anharmonic oscillator

$$H = p^2 + x^{2K}(ix)^\epsilon \quad (\epsilon > 0, \ K = 1, 2, 3, \cdots). \tag{1}$$

are discrete and positive and real even though H is non-Hermitian $H^\dagger \neq H$ in the conventional sense, where $H^\dagger$ denotes complex conjugate transpose. Even though conventional Hermiticity has been replaced by the weaker condition of $\mathcal{PT}$ symmetry, the Hamiltonian H defines a consistent quantum theory involving an unconventional scalar product [5]. The Hamiltonian operator H is defined by the Schrödinger equation

$$-\psi_n''(x) + [x^{2K}(ix)^\epsilon - E_n]\psi_n(x) = 0. \tag{2}$$

The eigenfunction $\psi_n(x)$ satisfies the $\mathcal{PT}$-symmetric boundary conditions that $\psi_n(x)$ vanishes as $|x| \to \infty$ in two wedges symmetrically placed about the imaginary axis in the lower-half x plane. As explained in [6], the wedges are determined by analytic continuation based on the leading-order exponentials in the asymptotic solutions of (2), namely

$$\psi(x) \sim \exp\left(\pm\frac{i^{\epsilon/2}x^{K+1+\epsilon/2}}{K+1+\epsilon/2}\right). \tag{3}$$

Within each wedge, one of the two solutions decays and one grows. Thus, the wedges are centred on the asymptotic directions

$$\theta_{\text{right}} = -\frac{\epsilon\pi}{4K + 2\epsilon + 4}, \qquad \theta_{\text{left}} = -\pi + \frac{\epsilon\pi}{4K + 2\epsilon + 4}. \tag{4}$$

The exponents in these asymptotic exponentials are purely real on the Stokes lines [7] at the centres of the wedges. It is easy to check that

in the right wedge: ψ_- decays and ψ_+ grows;

in the left wedge: $\begin{cases} \psi_- \text{ decays and } \psi_+ \text{ grows if } K \text{ is odd}, \\ \psi_+ \text{ decays and } \psi_- \text{ grows if } K \text{ is even}. \end{cases}$ (5)

The opening angle of each wedge is

$$\theta_{\text{opening angle}} = \frac{2\pi}{2K + \epsilon + 2}. \tag{6}$$

The wedge boundaries are anti-Stokes lines [7], where the solutions are purely oscillatory [1].

We are concerned here with the infinite subclass $\epsilon = 2$ for which the potential in (1), that is

$$V(x) = -x^{2k} \quad (K = 1, 2, 3, \cdots), \tag{7}$$

is real and appears to have the wrong sign to possess bound states. However, the $\mathcal{PT}$-symmetric solution to the Schrödinger equation (2) with the complex boundary conditions described above does have bound states.

Notice that when $\epsilon = 2$, the upper edges of the right and left wedges defined above (where the solutions are purely oscillatory) lie exactly along the positive and negative real-x axis. On the real axis the potential (7), when interpreted in conventional terms, describes one-dimensional scattering solutions of the Schrödinger equation, that is, travelling waves rather than decaying exponentials at infinity. Our central point here is that as far as the energy spectrum is concerned, we may replace the non-Hermitian eigenvalue problem in the complex wedges by a conventional Hermitian problem defined by the requirement that the potential be reflectionless. (To be precise, we are *not* claiming that these two quantum theories, the non-Hermitian $\mathcal{PT}$-symmetric theory defined in the complex wedges and the Hermitian theory defined on the real axis, are the same. However, these two distinct theories do have the same energy spectrum and eigenfunctions.)

To justify the assertion that we may replace the $\mathcal{PT}$-symmetric theory by the Hermitian one, consider the two dominant exponentials (3) when $\epsilon = 2$:

$$\psi_\pm(x) \sim \exp\left(\pm i\frac{x^{K+2}}{K+2}\right). \tag{8}$$

For real x these behaviours represent waves travelling in directions given by the sign of the current $\text{Im}\left(\psi_\pm^* \psi_\pm'\right)$, which is proportional to $\pm x^{K+1}$. Thus,

when $x > 0$:

$\psi_+(x)$ travels to the right and $\psi_-(x)$ travels to the left;

when $x < 0$:

$$\begin{cases} \psi_+(x) \text{ travels to the right and } \psi_-(x) \text{ travels to the left } (K \text{ odd}), \\ \psi_-(x) \text{ travels to the right and } \psi_+(x) \text{ travels to the left } (K \text{ even}). \end{cases} \quad (9)$$

These conditions match those for decay and growth in (5), so decay in the non-Hermitian problem corresponds to a purely left-travelling wave that is reflectionless in the corresponding conventional Hermitian problem. Under $\mathcal{PT}$ reflection, that is, replacing $x \to -x$ and $i \to -i$ in (8), both wave directions reverse if K is even and do not reverse if K is odd. Reflectionlessness persists in both cases.

The corresponding energies can be approximated by the WKB method, starting with identification of the two turning points, defined by $V(x) = E$, in the wedges described above. The turning points are

$$x_{\text{right}} = E^{1/(2K+2)} e^{-i\pi/(2K+2)} \quad \text{and} \quad x_{\text{left}} = E^{1/(2K+2)} e^{-i\pi + i\pi/(2K+2)}. \quad (10)$$

Quantization for large n according to

$$\int_{x_{\text{left}}}^{x_{\text{right}}} dt \sqrt{E - V(t)} = \left(n + \frac{1}{2}\right) \pi \quad (11)$$

leads straightforwardly [8] to

$$E_n \sim \left(\frac{(n+1/2)\sqrt{\pi}(K+2)\Gamma[(K+2)/(2K+2)]}{\Gamma[1/(2K+2)]\cos[\pi/(2K+2)]}\right)^{(2K+2)/(K+2)}. \quad (12)$$

Reflectionlessness can be regarded as the consequence of destructive interference between exponentially small waves reflected separately from the turning points x_{right} and x_{left} in (10). A similar interference phenomenon occurs in the analogous problem of exponentially weak adiabatic transition probabilities, which can also vanish if there are interfering contributions from complex turning points [9]. In the present example, as in some others considered recently [10], the vanishing reflection requires more than one pair of complex turning points and the corresponding potentials have vanishing curvature at the top of the potential at $x = 0$.

Reflectionlessness can also arise for other reasons. With potential barriers of finite range, for example, the interfering contributions come from nonanalytic extremities of the potential [11]. Alternatively, the cancellation can arise from a cluster of turning points which (unlike those considered here) cannot be separated [12], in which case reflectionlessness can persist for a continuum of parameter values rather than for isolated energies, such as those in (12).

In conclusion, we have shown that for an infinite class of non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians having unbroken $\mathcal{PT}$ symmetry, and thus real positive discrete spectra, there is a corresponding set of Hermitian Hamiltonians whose spectra and eigenfunctions become identical when the condition of reflectionlessness is imposed. Although these pairs of Hamiltonians do not describe the same physics because the inner product needed to calculate matrix elements is different, this intriguing association between non-Hermitian and Hermitian Hamiltonians may help to explain the surprising

observation [1] that the spectra of some non-Hermitian Hamiltonians can be real. We do not yet know if the association described in this paper extends to more general classes of $\mathcal{PT}$-symmetric Hamiltonians.

The connection between reflectionless potentials and $\mathcal{PT}$ symmetry may find application in quantum cosmology. Recently, much attention has been given to Anti-de Sitter cosmologies [13] and de Sitter cosmologies [14, 15]. In the AdS description the universe propagates reflectionlessly in the presence of a wrong-sign potential ($-x^6$, for example). In the dS case the usual Hermitian quantum mechanics must be abandoned and be replaced by a non-Hermitian one in which there are 'meta-observables'. The non-Hermitian inner product that is used in the dS case is based on the $\mathcal{CPT}$ theorem in the same way that the $\mathcal{CPT}$ inner product is used in $\mathcal{PT}$-symmetric quantum theory [5].

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